Holomorphic quantum stochastic cocycles
&
dilation of minimal quantum dynamical semigroups

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Outline

Quantum dynamical semigroups (QDS)

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for a Hilbert space k *and operators* $K \in B(\mathfrak{h})$ *and* $L \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$.

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Then there is a unique minimal QDS $\mathcal{T}^{K,L}$ *associated to* (K, L).
If $\mathcal{T}^{K,L}$ *is conservative then* $\mathcal{L}_{(K,L)}(1) = 0$*, in other words*

$$\|Lv\|^2 + 2\,\mathrm{Re}\langle v, Kv\rangle = 0, \quad v \in \mathrm{Dom}\, K.$$

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$\mathcal{F} = \mathcal{F}_{[0,r[} \otimes \mathcal{F}_{[r,t[} \otimes \mathcal{F}_{[t,\infty[}$, where $\mathcal{F}_{[r,t[} := \Gamma(L^2([r,t[; \mathsf{k}))$

Quantum stochastic (QS) contraction cocycles on $\mathfrak{h}$

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$V = (V_t)_{t\geq 0}$ contractions in $B(\mathfrak{h} \otimes \mathcal{F})$ satisfying

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Let $V \in \mathbb{QS}_c\mathbb{C}(\mathfrak{h}, \mathsf{k})$ be nonsingular. Then $\widetilde{V}$ is nonsingular and, with $K^V := K_{0,0}^V$, $L^V := L_0^V$ and $\widetilde{L}^V := L_0^{\widetilde{V}}$,

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$V \in \mathbb{QS}_c\mathbb{C}(\mathfrak{h}, \mathsf{k})$ is *Markov-regular* if its expectation semigroup is norm-continuous. Write $\mathbb{QS}_c\mathbb{C}_{\mathrm{M.reg}}(\mathfrak{h}, \mathsf{k})$ for this class.

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Then the QSDE $dV_t = V_t d\Lambda_F(t)$, $V_0 = I$ has a unique (strong) solution. Notation: V^F.

Bounded QS generators

$$C_0(\mathfrak{h}, \mathsf{k}) := \{F \in B(\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k})) : r(F) \leq 0\},$$

$r(f) := F^* + F + F^*\Delta F \leq 0 \leq 0$ iff $q(F) := F + F^* + F\Delta F^* \leq 0$.

Theorem

The map $F \mapsto V^F$ restricts to a bijection

$$C_0(\mathfrak{h}, \mathsf{k}) \to \mathbb{QS}_c\mathbb{C}_{\mathrm{M.reg}}(\mathfrak{h}, \mathsf{k}).$$

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Theorem (Ouhabaz)

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$$P_t v = \lim_{n\to\infty} (I - n^{-1}tG)^{-n} v \quad (v \in \mathfrak{h}),$$

$$\mathcal{Q} = \left\{ v \in \mathfrak{h} : \sup_{t>0} t^{-1} \operatorname{Re}\langle v, (I - P_t)v\rangle < \infty \right\}$$

$$q[v] = \lim_{t\to 0^+} t^{-1}\langle v, (I - P_t)v\rangle$$

$$\operatorname{Dom} G = \{v \in \mathcal{Q} : \exists_{v'\in\mathfrak{h}} \forall_{u\in\mathcal{Q}} \ \langle u, v'\rangle = -q(u, v)\}, \ \ Gv = v'.$$

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where $\mathcal{Q}$ is the domain of the quadratic form associated with K.

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$\widetilde{V}$ is homolorphic if and only if V is.

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$$\operatorname{Dom} \Gamma = \operatorname{Dom} \Delta F = \mathcal{Q} \oplus (\mathfrak{h} \otimes \mathsf{k}),$$

$$\Gamma[\zeta] = \gamma[v] - \{\langle \xi, Lv \rangle + \langle \widetilde{L}v, \xi \rangle + \langle \xi, (C - I)\xi \rangle\} \text{ for } \zeta = \begin{pmatrix} v \\ \xi \end{pmatrix},$$

$$\Delta F = \begin{bmatrix} 0 & 0 \\ L & C - I \end{bmatrix}.$$

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In particular, $(K, L, -L, 0)$, $(K, L, 0, -I) \in \mathfrak{X}_4^{\mathrm{Hol}}(\mathfrak{h},\mathsf{k})$.

The stochastic generator of a homomorphic QS cocycle

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The prescription

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Definition

For $V \in \mathbb{QS}_c\mathbb{C}_{\mathrm{Hol}}(\mathfrak{h}, \mathsf{k})$,
we refer to $\mathbb{F}^V$ as the *stochastic generator* of V.

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where $(K, L) \in \mathfrak{X}_2^{\mathrm{Hol}}(\mathfrak{h}, \mathsf{k})$ *is the truncation of the stochastic generator of* V *to its first two components*

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Let $V \in \mathbb{QS}_c\mathbb{C}_{\mathrm{Hol}}(\mathfrak{h}, \mathsf{k})$. *Then*

$$\mathbb{E}_0\big[V_t^*(x \otimes I_{\mathcal{F}})V_t\big] = \mathcal{T}_t^{K,L}(x), \quad x \in B(\mathfrak{h}), t \geq 0. \qquad (2)$$

where $(K, L) \in \mathfrak{X}_2^{\mathrm{Hol}}(\mathfrak{h}, \mathsf{k})$ *is the truncation of the stochastic generator of* V *to its first two components* [*i.e.* $\mathbb{F}^V$ *is of the form* $(K, L, *, *)$].

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Let $(K, L) \in \mathfrak{X}_2^{\mathrm{Hol}}(\mathfrak{h}, \mathsf{k})$. *Then, letting* $V = V^{\mathbb{F}}$, *where* $\mathbb{F} = (K, L, -C^*L, C - I)$ *for a contraction* $C \in B(\mathfrak{h} \otimes \mathsf{k})$,

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